ON THE "ZERO-TWO" LAW*

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ABSTRACT

Results of Ornstein-Sucheston, are extended to non-separable measure spaces and operators that are not induced by a transition probability.

Notation. We shall use the notation of [2]. Let (X,Σ,m,P) be a Markov process with $m(X) = 1$. Assume that P is ergodic and conservative: If $0 \leqq f \in L_\infty$ and $f \not\equiv 0$ *then* $\Sigma_{n=0} P^n f \equiv \infty$. Note that all inequalities employed are a.e. inequalities.

1. The sequence of suprema

Let us define:

$$h_n = \sup\{(P^n g - P^{n+1} g) \colon -1 \leqq g \leqq 1\}.$$

Note that the supremum is in the L_∞ sense as defined in [1,IV.11.7].

THEOREM 1.1. *The sequence h_n satisfies:*

(a) $0 \leqq h_n \leqq 2$

(b) $h_n \geqq h_{n+1}$

(c) $Ph_n \geqq h_{n+1}$

(d) $\lim h_n = \text{Const.}$

PROOF.

(a) $0 = P^n 0 - P^{n+1} 0 \leqq \sup\{(P^n g - P^{n+1} g) \colon -1 \leqq g < 1\} \leqq P^n 1 + P^{n+1} 1 \leqq 2.$

(b) $P^{n+1} g - P^{n+2} g = P^n(Pg) - P^{n+1}(Pg) \leqq h_n$ since $-1 \leqq Pg \leqq 1$ if $-1 \leqq g \leqq 1$.

(c) For every $-1 \leqq g \leqq 1$

$$Ph_n \geqq P(P^n g - P^{n+1} g) = P^{n+1} g - P^{n+2} g$$

and the supremum on the right hand side is h_{n+1}.

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* This research was supported in part by the National Bureau of Standards PL480 program grant NBS (g-130).

Received December 21, 1970

(d) Let $h = \lim h_n$, then $0 \leqq h \leqq 2$. Now $Ph = \lim Ph_n \geqq \lim h_{n+1} = h$ and by [2, chap. II, th. B] $h = \text{Const.}$

Let us denote the above constant by α.

COROLLARY. *$0 \leqq \alpha \leqq 2$ and $\alpha = 2$ if and only if $h_n(x) = 2$ for all x and n.*

As in [2, p. 54], we define $U_n = \inf\{P^n, P^{n+1}\}$ by

$$0 \leqq f \in L_\infty : U_n f = \inf\{P^{n+1}f' + P^n(f - f') : 0 \leqq f' \leqq f\}.$$

THEOREM 1.2. $U_n 1 = 1 - \frac{1}{2}h_n$.

PROOF. $f = \frac{1}{2}(g+1)$ sends the family $-1 \leqq g \leqq 1$ onto $0 \leqq f \leqq 1$. Thus $P^n f - P^{n+1} f = \frac{1}{2}(P^n g - P^{n+1} g)$ or

$$\begin{aligned} U_n 1 &= \inf\{1 - (P^n f - P^{n+1} f) : 0 \leqq f \leqq 1\} \\ &= 1 - \sup\{P^n f - P^{n+1} f) : 0 \leqq f \leqq 1\} = 1 - \tfrac{1}{2}h_n. \end{aligned}$$

COROLLARY. *$\alpha < 2$ if and only if there exists a Markov operator $Q \neq 0$ and an integer n such that $Q \leqq P^n$ and $Q \leqq P^{n+1}$.*

2. *If $\alpha < 2$ then $\alpha = 0$.* Throughout this section we will assume that $\alpha < 2$. For every integer k the operator P^k is conservative; see [3, cor. 2] On the other hand P^k need not be ergodic. Put $\Sigma_i(P^k) = \{A : P^k 1_A = 1_A\}$.

LEMMA 2.1. *The σ-field $\Sigma_i(P^k)$ is atomic. If $A \in \Sigma_i(P^k)$ is an atom then $P^i 1_A$ $0 \leqq i < k$ are characteristic functions of disjoint sets.*

PROOF. If $\Sigma_i(P^k)$ is non-atomic, let A_n be a decreasing sequence of sets in $\Sigma_i(P^k)$ with $m(A_n) \to 0$. Now

$$0 = (I - P^k)1_{A_n} = (I - P)(1_{A_n} + P1_{A_n} + \cdots + P^{k-1}1_{A_n}).$$

Thus $1_{A_n} + P1_{A_n} + \cdots + P^{k-1}1_{A_n} = \text{Const.}$ as P is ergodic, or

$$1_{A_n} + P1_{A_n} + \cdots + P^{k-1}1_{A_n} \geqq 1.$$

But as $n \to \infty$ each term tends to zero. Let now A be an atom of $\Sigma_i(P^k)$ then $P^i 1_A$ are characteristic functions, by [2, chap. III, th. A], of the sets A_i and $A_i \cap A_j \in \Sigma_i(P^k)$ too. Hence, the intersection is empty.

A more general result is obtained in [4, th. 1].

COROLLARY. *If $\alpha < 2$ then P^k is ergodic for every integer k.*

PROOF. Let A be as above, then $h_{nk} \geqq 2(P^{nk}1_A - P^{nk+1}1_A) = 2$ on A and $\alpha = 2$, too.

In the following construction we use the methods of [5].

Let us use the Corollary of Theorem 1.2 to write $P^k = Q + R$, $P^{k+1} = Q + S$, $Q \neq 0$, and Q, R, and S are Markov operators. Thus,

$$P^{k+1} = \tfrac{1}{2}Q(I + P) + \tfrac{1}{2}(RP + S) = \tfrac{1}{2}Q(I + P) + T$$

Also

$$\begin{aligned} T1 &= \tfrac{1}{2}(RP1 + S1) = \tfrac{1}{2}(R1 + S1) \\ &= \tfrac{1}{2}(P^k 1 - Q1) + \tfrac{1}{2}(P^{k+1}1 - Q1) = 1 - Q1 \neq 1. \end{aligned}$$

The sequence $T^n 1$ is monotone and its limit g satisfies $Tg = g$. Thus $P^{k+1}g \geqq g$ and since P^{k+1} is ergodic and conservative $P^{k+1}g = g$ and $g = \text{Const}$.

Thus, $Qg = \tfrac{1}{2}Q(I + P)g = (P^{k+1} - T)g = 0$. Hence, $g = 0$ also. We have proved:

LEMMA 2.2. *If $\alpha < 2$ then for some integer k $P^k = \tfrac{1}{2}Q(I + P) + T$ where Q and T are Markov operators and $T^n 1 \downarrow 0$.*

LEMMA 2.3. *Let $\alpha < 2$. For every integer n there exists an integer m and a Markov operator Q_m with $P^m = \tfrac{1}{2}Q_m(I + P) + T^n$.*

PROOF. The case $n = 1$ is proved in Lemma 2.2. Assume the Lemma for n.

$$\begin{aligned} P^m P^k &= \tfrac{1}{2}Q_m P^k(I + P) + T^n P^k \\ &= \tfrac{1}{2}Q_m P^k(I + P) + \tfrac{1}{2}T^n Q(I + P) + T^{n+1}. \end{aligned}$$

Note that Q_m is a bounded positive operator on L_1. Now

$$Q_m 1 = \tfrac{1}{2}Q_m(I + P)1 \leqq P^m 1 \leqq 1.$$

Hence Q_m is necessarily a Markov operator.

Let $\varepsilon > 0$ be given and choose $n = n_1$ so large that $m\{x\colon T^n 1(x) < \varepsilon\} > \tfrac{1}{2}$. Call the corresponding integer $m = m_1$:

$$\begin{aligned} P^{m_1} &= \tfrac{1}{2}Q_{m_1}(I + P) + T^n \\ P^{m_1 + m_2} &= \tfrac{1}{2}Q_{m_1}P^{m_2}(I + P) + T^n P^{m_2} \\ &= \tfrac{1}{2}Q_{m_1}[\tfrac{1}{2}Q_{m_2}(I + P) + T^{n_2}](I + P) + T^n P^{m_2} \\ &= \tfrac{1}{4}Q_{m_1}Q_{m_2}(I + P)^2 + \tfrac{1}{2}Q_{m_1}T^{n_2}(I + P) + T\ P^{m_2} \end{aligned}$$

Now put

$$\{x\colon \tfrac{1}{2}Q_{m_1}T^{n_2}(I + P)1(x) < \varepsilon/2\} = A_2$$

Choose n_2 (and thus m_2) so large that $m(A_2 \cap A_1) > \tfrac{1}{2}$.

In general after k steps we would have

$$P^{m_1+m_2+\cdots+m_k} = \frac{1}{2^k} Q_k^*(I+P)^k + T_k^*$$

where $T_k^*1(x) \leqq \varepsilon \sum_{i=0}^{k-1} 1/2^i$ on A_k and $m(A_1 \cap \cdots \cap A_k) > \frac{1}{2}$. Repeat the above argument

$$P^{m_1+\cdots+m_k+m_{k+1}} = \frac{1}{2^k} Q_k^* P^{m_{k+1}}(I+P)^k + T_k^* P^{m_{k+1}}$$

$$= \frac{1}{2^{k+1}} Q_k^* Q_{k+1}(I+P)^{k+1} + \frac{1}{2^k} Q_k^* T^{n_{k+1}}(I+P)^k$$

$$+ \; T_k^* P^{m_{k+1}}$$

and $A_{k+1} = \{x : Q_k^* T^{n_{k+1}} 1(x) < 1/2^n\}$ can be chosen so large that

$$m(A_1 \cap \cdots \cap A_{k+1}) > \tfrac{1}{2}.$$

Let us summarize:

LEMMA 2.4. *Let $\alpha < 2$. Given $\varepsilon > 0$, there exists a sequence n_k, Markov operators Q^* and T^* with*

$$P^{n_k} = Q_k^* \frac{1}{2^k}(I+P)^k + T_k^*; \; m \bigcap_{k=1}^{\infty} \left\{x : T_k^* 1(x) < \varepsilon \sum_{i=0}^{k-1} \frac{1}{2^i}\right\} > \tfrac{1}{2}.$$

PROOF. It is clear, from the construction that Q^*, T^* are bounded positive operators on L_1. Thus, it is enough to show that they are bounded by 1:

$$T_k^* 1 \leqq P^{n_k} 1 = 1, \; Q_k^* 1 = Q_k^* \frac{1}{2^k}(I+P)^k 1 \leqq P^{n_k} 1 = 1.$$

THEOREM 2.5. *If $\alpha < 2$, then $\alpha = 0$.*

PROOF. Put $B = \bigcap_{k=1}^{\infty}\{x : T_k^* 1(x) < 2\varepsilon\}$, then $m(B) > \frac{1}{2}$. Let $-1 \leqq g \leqq 1$.

$$P^{n_k}(I-P)g = Q_k^* \frac{1}{2^k}(I+P)^k(I-P)g + T_k^*(I-P)g.$$

On B $T_k^*(I-P)g \leqq 4\varepsilon$. Now on the other hand

$$\left| Q_k^* \frac{1}{2^k}(I+P)^k(I-P)g \right| = \left| Q_k^* \frac{1}{2^k} \sum_{j=1}^{k} \left(\binom{k}{j} - \binom{k}{j-1} \right) P^k g \right|$$

$$\leqq \frac{1}{2^k} \sum_{j=1}^{k} \left| \binom{k}{j} - \binom{k}{j-1} \right|$$

as in [5], 1.8 this tends to zero as $k \to \infty$.

REMARKS. By Theorems 1.2 we obtain:

If $\alpha < 2$ then $U_n 1 \uparrow 1$. Let $\alpha < 2$ and $0 \leqq u \in L_1$ then $\| u(I-P)P^n \| = \langle u(I-P)P^n, g \rangle$ for some $-1 \leqq g \leqq 1$ so $\| u(I-P)P^n \| \leqq \langle u, h_n \rangle \to 0$. Hence, *if* $\alpha < 2\,(\alpha = 0)$ *and* $u \in L_1$ *then* $\| u(I-P)P^n \| \to 0$. Since $\overline{L_1(I-P)} = \{v : \int v dm = 0\}$ (as $(I-P)f = 0$ if and only if f is a constant) one obtains:

COROLLARY. *Let* $\alpha < 2$. *If* $v \in L$, *and* $\int v dm = 0$ *then* $\| vP^n \|_1 \to 0$.

AN EXAMPLE. Let $P = \delta P_1 + (1-\delta)P_2$ $0 < \delta < 1$ and P is ergodic and conservative. (If $P_1 = I$, P is ergodic and conservative provided P_2 is). Now $P^n \geqq \delta^n P_1^n$, $P^{n+1} \geqq \delta^{n+1} P_1^{n+1}$ thus if $\alpha(P_1) = 0$, then $\inf(P_1^n, P^{n+1}) \neq 0$ for some n and therefore $\inf(P^n, P^{n+1}) \neq 0$ and $\alpha(P) = 0$ too, by the Corollary to Theorem 1.2.

3. The case $d = 2$

Throughout this section, we shall assume that $\alpha = 2$ or $\inf(P^n, P^{n+1}) = 0$ for every n. If P is induced by a transition probability then for every $-1 \leqq g \leqq 1$, $(P^n g - P^{n+1} g)(x) \leqq \| P^n(x, \cdot) - P^{n+1}(x, \cdot) \| \leqq 2$. Since the L_∞ supremum of the left hand side is 2 we obtain:

THEOREM 3.1. *If* $\alpha = 2$, *then* $\| P^n(x, \cdot) - P^{n+1}(x, \cdot) \| = 2$ a.e.

Our assumption is that for every n $\sup\{(P^n - P^{n+1})g : -1 \leqq g \leqq 1\} = 2$. Now, according to [1, IV,11.7] a countable union will suffice. Thus there exists a sequence $-1 \leqq g_k \leqq 1$ and sets $B_{k,N}$ where $B_{1,N}, \cdots, B_{N,N}$ are disjoint such that

$$\sum_{k=1}^{N} 1_{B_{k,N}} (P^n - P^{n+1}) g_k \to 2$$

or for every $\varepsilon > 0$

$$X = \cup_k \{x : (P^n - P^{n+1}) g_k(x) > 2 - \varepsilon\}.$$

Let $g_k = g_k^+ - g_k^-$, $g_k^+ \leqq 1_{A_k}$, $g^- \leqq 1_{A'_k}$ $(A_k = \{x : g_k(x) > 0\})$ $(P^n - P^{n+1}) g_k(x) = P^n g_k^+(x) + P^{n+1} g_k^-(x) - P^n g_k^-(x) - P^{n+1} g_k^+(x)$ $\{x : (P^n - P^{n+1}) g_k(x) \geqq 2 - \varepsilon\} \subset \{x : P^n g_k^+(x) \geqq 1 - \varepsilon\} \cap \{x : P^{n+1} g_k^-(x) \geqq 1 - \varepsilon\}$ Or on this set:

$$(P^n - P^{n+1})(1_{A_k} - 1_{A'_k}) \geqq 2 - 4\varepsilon$$

Thus $\sup\{(P^n - P^{n+1})(1_A - 1_{A'}) : A \in \Sigma\} = 2$. (Again, we can take the supremum of a countable collection of sets.)

Clearly, the above implies, by replacing $\phi = 1_A - 1_{A'}$, by $(\phi + 1)/2 = 1_A$, that $\sup\{(P^n - P^{n+1}) 1_A : A \in \Sigma\} = 1$.

Let us assume that P is induced by a transition probability. In [2, chap. V, (5.1)] it is proved that $P^n[A \times B] = \int_B P^n 1_A dm$ extends to a σ-additive measure on $\Sigma \times \Sigma$. Now $\sum_{k=1}^{N} 1_{B_{k,N}} (P^n - P^{n+1}) g_k \to 2$, integrate to get:

$$(\tilde{P}^n - \tilde{P}^{n+1}) \left(\sum_{k=1}^{N} g_k(x) 1_{B_{k,N}}(y) \right) \to 2.$$

Since $B_{k\ N}$ are disjoint sets $\left| \Sigma_{k=1}^{N} g_k(x) 1_{B_{k,N}} \right| \leqq 1$. Thus, $\| \tilde{P}^n - \tilde{P}^{n+1} \| = 2$ (the norm of the measure). Thus

THEOREM 3.2. *Let P be induced by a transition probability. The following conditions are equivalent:*

(a) $\alpha = 2$

(b) $\tilde{P}^n \perp \tilde{P}^{n+1}$ *for every n.*

PROOF. If $\alpha = 2$, then $\| \tilde{P}^n - \tilde{P}^{n+1} \| = 2$ so the measures are singular. If $\alpha < 2$, then $\inf(P^n, P^{n+1}) = U_n \neq 0$ and $\tilde{U}_n \leqq \tilde{P}^n$ $\tilde{U}_n \leqq \tilde{P}^{n+1}$; so $\tilde{P}^n$ and $\tilde{P}^{n+1}$ are not singular.

If there exists a set A such that $P^n 1_A$ is a characteristic function for every n (for instance if P is induced by a point transformation) then $(P^n - P^{n+1})1_A$ can assume the values 1, 0, -1 only. If it assumes the value 1 then $\alpha = 2$. If it assumes the value -1 apply $P^n - P^{n+1}$ to $1_{A'}$ to get again $\alpha = 2$. Finally, if $P^n 1_A = P^{n+1} 1_{A'}$ $P^n 1_A = \text{Const.}$ since P is ergodic and conservative. Thus, $P^n 1_A = 1 \geqq 1_A$ so $P^n 1_A = 1_A$ or $A = X$ necessarily. Therefore, if $P^n 1_A$ is a characteristic function for every n where $A \neq \varnothing$, $A \neq X$, then $\alpha = 2$.

REFERENCES

1. N. Dunford and J. T. Schwartz, *Linear operators, Part I,* Intersciences publishers, New York, 1958.

2. S. R. Foguel, *The ergodic theory of Markov processes,* Van Nostrand, New York, 1969.

3. S. R. Foguel, *Remarks on conservative Markov processes,* Israel J. Math. **6** (1968), 381–383.

4. S. T. C. Moy, *Period of an irreducible operator,* Illinois J. Math. **11** (1967), 24–39.

5. D. Ornstein and L. Sucheston, *An operator theorem on L_1 convergence to zero,* Ann. Math. Statist. (1970).

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